Problem 2.34

The hyperbolic function $\tanh z$ is defined as $\tanh z = \sinh z / \cosh z$, with $\cosh z$ and $\sinh z$ defined as in Problem 2.33. **(a)** Prove that $\tanh z = -i \tan(iz)$. **(b)** What is the derivative of $\tanh z$? **(c)** Show that $\int dz \, \tanh z = \ln \cosh z$. **(d)** Prove that $1 - \tanh^2 z = \operatorname{sech}^2 z$, where $\operatorname{sech} z = 1/\cosh z$. **(e)** Show that $\int dx/(1 - x^2) = \operatorname{arctanh} x$.

[TYPO: Replace $\ln \cosh z$ with $\ln \cosh z + C$, and replace $\operatorname{arctanh} x$ with $\operatorname{arctanh} x + C$.]

Solution

The definitions of $\sinh z$ and $\cosh z$ are given in Problem 2.33.

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

As a result, the hyperbolic tangent function is defined by

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z} \\ &= \frac{\frac{e^z - e^{-z}}{2}}{\frac{e^z + e^{-z}}{2}} \\ &= \frac{e^z - e^{-z}}{e^z + e^{-z}}. \end{aligned}$$

Part (a)

Sine and cosine are defined in terms of exponential functions by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The tangent function is the ratio of the sine and cosine functions.

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} \\ &= \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} \\ &= \frac{1}{i}\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right) \end{aligned}$$

Replace z with iz in the formula for tangent.

$$\tan iz = \frac{1}{i}\left[\frac{e^{i(iz)} - e^{-i(iz)}}{e^{i(iz)} + e^{-i(iz)}}\right]$$

$$= \frac{1}{i}\left(\frac{e^{-z} - e^{z}}{e^{-z} + e^{z}}\right)$$

$$= -\frac{1}{i}\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right)$$

$$= -\frac{1}{i}\tanh z$$

Therefore, multiplying both sides by $-i$,

$$\tanh z = -i \tan iz.$$

Part (b)

Take the derivative of the hyperbolic tangent function.

$$\frac{d}{dz}\tanh z = \frac{d}{dz}\left(\frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}\right)$$

$$= \frac{\left[\frac{d}{dz}(e^{z} - e^{-z})\right](e^{z} + e^{-z}) - \left[\frac{d}{dz}(e^{z} + e^{-z})\right](e^{z} - e^{-z})}{(e^{z} + e^{-z})^2}$$

$$= \frac{(e^{z} + e^{-z})(e^{z} + e^{-z}) - (e^{z} - e^{-z})(e^{z} - e^{-z})}{(e^{z} + e^{-z})^2}$$

$$= \frac{(e^{2z} + 1 + 1 + e^{-2z}) - (e^{2z} - 1 - 1 + e^{-2z})}{(e^{z} + e^{-z})^2}$$

$$= \frac{4}{(e^{z} + e^{-z})^2}$$

$$= \frac{1}{\left(\frac{e^{z} + e^{-z}}{2}\right)^2}$$

$$= \frac{1}{\cosh^2 z}$$

$$= \operatorname{sech}^2 z$$

Part (c)

Take the integral of the hyperbolic tangent function.

$$\begin{aligned}
\int \tanh z\, dz &= \int^z \tanh z'\, dz' + C \\
&= \int^z \frac{\sinh z'}{\cosh z'}\, dz' + C \\
&= \int^z \frac{\frac{e^{z'} - e^{-z'}}{2}}{\frac{e^{z'} + e^{-z'}}{2}}\, dz' + C \\
&= \int^z \frac{\frac{d}{dz'}\left(\frac{e^{z'} + e^{-z'}}{2}\right)}{\frac{e^{z'} + e^{-z'}}{2}}\, dz' + C \\
&= \int^z \frac{\frac{d}{dz'}(\cosh z')}{\cosh z'}\, dz' + C \\
&= \int^z \frac{d}{dz'}\left(\ln \cosh z'\right) dz' + C \\
&= \left. \ln \cosh z' \right|^z + C \\
&= \ln \cosh z + C
\end{aligned}$$

Part (d)

Simplify the given expression.

$$\begin{aligned}
1 - \tanh^2 z &= (1 + \tanh z)(1 - \tanh z) \\
&= \left(1 + \frac{e^z - e^{-z}}{e^z + e^{-z}}\right)\left(1 - \frac{e^z - e^{-z}}{e^z + e^{-z}}\right) \\
&= \left(\frac{e^z + e^{-z}}{e^z + e^{-z}} + \frac{e^z - e^{-z}}{e^z + e^{-z}}\right)\left(\frac{e^z + e^{-z}}{e^z + e^{-z}} - \frac{e^z - e^{-z}}{e^z + e^{-z}}\right) \\
&= \left[\frac{(e^z + e^{-z}) + (e^z - e^{-z})}{e^z + e^{-z}}\right]\left[\frac{(e^z + e^{-z}) - (e^z - e^{-z})}{e^z + e^{-z}}\right] \\
&= \left(\frac{2e^z}{e^z + e^{-z}}\right)\left(\frac{2e^{-z}}{e^z + e^{-z}}\right) \\
&= \frac{4}{(e^z + e^{-z})^2}
\end{aligned}$$

Therefore,

$$1 - \tanh^2 z = \frac{1}{\left(\frac{e^z + e^{-z}}{2}\right)^2}$$

$$= \frac{1}{\cosh^2 z}$$

$$= \operatorname{sech}^2 z.$$

Part (e)

Evaluate the given integral.

$$\int \frac{dx}{1 - x^2} = \int^x \frac{dx'}{1 - x'^2} + C$$

Make the following substitution.

$$x' = \tanh z$$

$$dx' = \operatorname{sech}^2 z \, dz$$

As a result,

$$\int \frac{dx}{1 - x^2} = \int^{\tanh^{-1} x} \frac{\operatorname{sech}^2 z \, dz}{1 - \tanh^2 z} + C$$

$$= \int^{\tanh^{-1} x} \frac{\operatorname{sech}^2 z \, dz}{\operatorname{sech}^2 z} + C$$

$$= \int^{\tanh^{-1} x} dz + C$$

$$= z \Big|^{\tanh^{-1} x} + C$$

$$= \tanh^{-1} x + C.$$